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Instability of non-commutative SYM theories at finite temperature

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Abstract: We extend our previous work on the quasi-particle excitations in $\mathcal{N} = 4$ non-commutative U(1) Yang-Mills theory at finite temperature. We show that above some critical temperature there is a tachyon in the spectrum of excitations. It is a collective transverse photon mode polarized in the non-commutative plane. Thus the theory seems to undergo a phase transition at high temperature. Furthermore we find that the group velocity of quasi-particles generically exceeds the speed of light at low momentum.

Keywords: Non-Commutative Geometry, Thermal Field Theory.

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Contents

1. Introduction

Quantum field theories on non-commutative spaces exhibit the intriguing phenomenon of UV/IR mixing [1]. On a technical level this arises as follows. Like in large-N gauge theories, one distinguishes between planar and non-planar graphs. Planar graphs have the usual UV behaviour of ordinary quantum field theories because the phases stemming from the Moyal product do not depend on the internal momenta. In non-planar graphs the phases do depend on the internal momenta [2], leading in general to the regularization of potentially divergent integrals. This regularization depends however on the inflowing momentum. The UV divergence reappears in the disguise of an IR divergence as the inflowing momentum goes to zero. This implies that high momentum modes do not decouple from the physics at large distances and furthermore makes renormalization of non-commutative field theories a difficult issue.

Depending on the details of the theory, UV/IR mixing may also lead to the appearance of tachyonic modes. This has been shown already in [1], in the case of a $\lambda\phi^3$ theory in six dimensions. Because the potential is unbounded from below, this theory is expected to be unstable. However, in the ordinary case the instability is non perturbative in a weak coupling expansion, while in the non-commutative case an instability appears already at one loop. Another example of a theory that can be unstable at one loop due to UV/IR mixing is the non-commutative version of a U(1) gauge theory. The photon polarization tensor has been first calculated in [3, 4], where it was shown that it acquires generically a pole-like IR divergent piece. The nature of the IR pole depends on the content of adjoint matter. It is remarkable that, if there are more bosonic than fermionic fields in the adjoint representation, the IR pole leads to the appearance of tachyonic modes at long wavelengths. In particular, tachyonic modes appear at one-loop level for pure U(1) gauge theory. In [5] it was shown that the tachyonic character of the IR pole is gauge independent. As of today, a satisfactory understanding of the effects of UV/IR mixing in general and of the existence of instabilities in particular is still missing.

Pole-like infrared divergences are absent in supersymmetric theories [4], and thus the implications of UV/IR mixing are somewhat less severe. Indeed, it has been argued that UV/IR mixing effects in logarithmic divergences do not threaten renormalizability [6, 7]. In four space-time dimensions, the maximally supersymmetric theory is $\mathcal{N}=4$ Yang-Mills theory. The commutative version is known to be UV finite and therefore one expects the non-commutative theory to be free of both pole-like and logarithmic IR divergences. This has been shown at one-loop level in [8]. Non-commutative $\mathcal{N}=4$ Yang-Mills theory is also interesting from the point of view of string theory. It arises as the effective low energy theory describing the physics of D3-branes in IIB string theory in the background of a magnetic B-field [9]. Many features of non-commutative gauge theories, like for instance non-commutative solitons, have a natural D-brane interpretation. (See e.g. [10] for a review.)

Since perturbation theory seems well-defined in $\mathcal{N}=4$ non-commutative Yang-Mills theory it is natural to ask what happens when supersymmetry is broken. One possible way to investigate this problem is by introducing temperature. Non-commutative theories at finite temperature have already been considered in a number of earlier works [11, 12]. In particular, in [12] we argued that introducing finite temperature allows to study the issue of UV/IR mixing in a controllable setting. Of course, conventional Yang-Mills field theories at finite temperature have their own share of IR troubles. (See e.g. [13] or [14].) However, if one considers non-commutative theories with Moyal bracket interactions, these specific problems are absent, simply because interactions switch-off for vanishing momentum. Therefore, it seems that $\mathcal{N}=4$ non-commutative U(1) at finite temperature is an interesting starting point to study the issue of UV/IR mixing.

In this paper we will extend our previous work on scalar dispersion relations in supersymmetric, non-commutative field theories. Specifically we will investigate the dispersion relation of vector fields and fermions in $\mathcal{N}=4$ non-commutative U(1) gauge theory at the one-loop level. The work is organized as follows. In section 2 we review our previous findings on scalar dispersion relations. We discuss the appearance of superluminous wave propagation at the lower end of the spectrum. We discuss the issue of causality and the relation to the string picture where the non-commutative theory appears in the Seiberg-Witten decoupling limit of a D3-brane in a background magnetic field.

Section 3 is the main body of the paper. We compute the polarization tensor of the vector bosons in the so-called hard thermal loop approximation (HTL) [15]. Compared to conventional gauge theories at finite temperature, $\mathcal{N}=4$ non-commutative U(1) presents a number of remarkable features. We show that a longitudinal collective excitation (the non-commutative relative of the plasmon mode of standard gauge theories) appears above a certain critical temperature $T_{\rm c}$. The existence of a critical temperature is a new feature which can be understood intuitively by picturing non-commutative particles in the adjoint as rigid dipoles [16]. The presence of a thermal bath also affects the transverse modes. In our setup we have to distinguish between transverse states polarized, respectively in the commutative and non-commutative directions. Our most striking result is that, above $T_{\rm c}$, there is a new collective transverse mode polarized in the non-commutative plane of tachyonic nature. We observe that, as $T \to \infty$ and in the static limit, the dispersion relations we find reproduce the pole-like IR divergence of the reduced three-dimensional non-supersymmetric theory which is the high temperature limit of the $\mathcal{N}=4$ theory.

In section 4 we complete our work by computing the self-energy of the fermions. At finite temperature generically we expect collective fermion excitations. Again we find that these are present in the model only for high temperatures. The critical temperature for the existence or these plasmino modes turns out to be higher than for the collective vector modes.

In section 5 we discuss our results and speculate on the physical implications of the tachyon. In two appendices we have collected the expressions for the polarization tensor and the fermion self-energy that result from the expansion of the Moyal phases, and give a short discussion of gauge independence of the two-point functions in the HTL approximation.

Finally let us set the stage for our calculation by introducing some conventions. Non-commutativity of the coordinates takes the form

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,. \tag{1.1}$$

Time is taken to be an ordinary commuting coordinate and $\theta^{\mu\nu}$ is an x-independent matrix. The algebra of functions is defined by the Moyal product

$$f(x) * g(x) = \lim_{y\to x} e^{i\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu} f(x)g(y)\,. \tag{1.2}$$

Star product commutators or so-called Moyal brackets are defined as $\{f,g\}_*(x) = f(x)*g(x) - g(x)*f(x)$. The non-commutative field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\{A_\mu, A_\nu\}_*$. The perturbative expansion is most easily derived by considering a ten-dimensional $\mathcal{N}=1$ theory

$$S = \frac{1}{2g^2}\int d^{10}x \left(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \bar\psi\Gamma^\mu\left(i\partial_\mu\psi + \{A_\mu,\psi\}_*\right)\right), \tag{1.3}$$

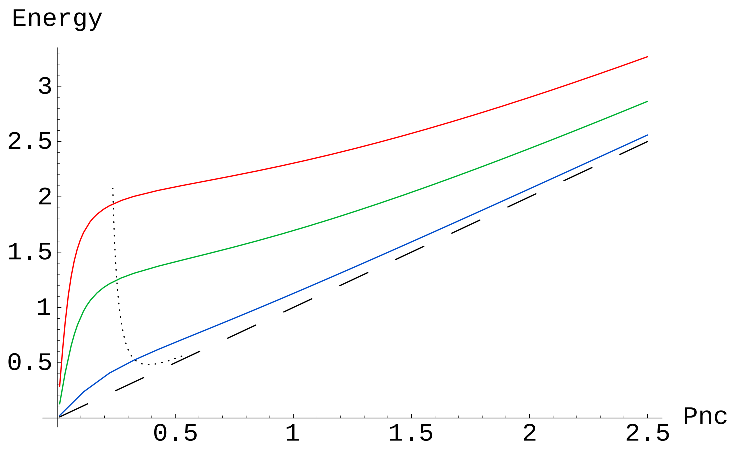

Figure 1: Dispersion relation for scalars in $\mathcal{N} = 4$ Yang-Mills for different temperatures. The momentum is taken to lie entirely in the non-commutative directions. The dashed line shows the light-cone. The dotted line shows the momentum below which the group velocity is bigger than one.

and restricting the momenta to four dimension. We will use capital letters to refer to four-momentum. In euclidean signature we will have $p_0 = i\omega$. Contraction of a vector with $\theta^{\mu\nu}$ will be denoted by $\tilde{p}^\mu = \theta^{\mu\nu} p_\nu$ and spatial momenta pointing in the non-commuting directions noted $p_{\rm nc}$. Finally, without loss of generality we will assume $\theta^{23} = \theta$, with all other components set to zero. The calculations are done using the Feynman gauge. (See also appendix B.)

2. Superluminal propagation

We first review the results of [12] on the dispersion relation of scalar excitations in non-commutative $\mathcal{N} = 4$ U(1) gauge theory at finite temperature. The scalar self-energy at one loop can be written in the form

$$\Sigma = 32g^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2 \frac{\tilde{p}\cdot k}{2}}{k} \left(n_B(k) + n_F(k)\right) + g^2 P^2 \bar{\Sigma}\,. \tag{2.1}$$

Here n_B and n_F are, respectively the Bose-Einstein and Fermi-Dirac distributions. Since the scalar fields are massless at tree-level, the second term is $O(g^4)$ and can be dropped in the sequel. The self-energy (2.1) then leads to the following dispersion relation

$$p_0^2 = p^2 + 2g^2T^2 - \frac{4g^2T}{\pi|\tilde{p}|} \tanh \frac{\pi|\tilde{p}|T}{2}\,, \tag{2.2}$$

A plot of (2.2) for different temperatures is shown in figure 1.

For large $p_{\rm nc}$ the dispersion relation is that of a massive particle. The last term in (2.2) corresponding to the non-planar contribution to the self-energy is suppressed. The planar contribution gives a thermal mass to the scalar modes, $m_T^2 = 2g^2T^2$. This

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is like in ordinary field theories at finite temperature where particles are dressed by interactions with the thermal bath and (usually) get a temperature dependent mass. At small $p_{\rm nc}$ however, the planar and non-planar contributions tend to cancel each other. This is a reflection of the vanishing of the couplings as $p_{\rm nc}$ goes to zero. It might be useful to think of the non-commutative particles as dipoles or rigid rods in the non-commutative plane, of size $l \sim \theta p_{\rm nc}$, orthogonal to $p_{\rm nc}$ and with only the endpoints interacting [16]. As $p_{\rm nc} \to 0$, the dipoles are non interacting. It immediately follows from these considerations that there must be a domain of $p_{\rm nc}$ with group velocity $\partial p_0/\partial p > 1$. We expect this result to be true to all orders in the weak coupling expansion.

In figure 1, the region corresponding to $\partial p_0/\partial p > 1$ is to the left of the dotted line. For small momenta lying entirely on the non-commutative plane, $p_0 = c_0 p - \gamma p^3 + O(p^5)$ with $c_0 = \sqrt{1 + \frac{g^2\pi^2\theta^2 T^4}{6}}$ and $\gamma = \frac{g^2\pi^4\theta^4 T^6}{120 c_0}$. This is the dispersion relation of the linearized Korteweg-de Vries equation and it can be shown that the profile of a scalar disturbance travelling along the non-commutative directions is given in terms of an Airy function: $\frac{1}{2(2\gamma t)^{1/2}} Ai(\frac{x - c_0 t}{(3\gamma t)^{1/3}})$. As discussed in [12] the first crest of the wave-train is well defined outside the light-cone for large times $t > \sqrt{\gamma/(c_0-1)^3}$.

In [12], we argued that superluminous wave propagation does not *a priori* violate causality because of the existence of a preferred class of reference frames where time can be considered as a commutative coordinate. Of course, the reason why group velocities larger than one are possible is that the theory is not Lorentz invariant. However this is not seen at tree level in perturbation theory since the free part of a non-commutative field theory does not differ from its commutative counterpart. The violation of Lorentz symmetry might be more directly noticeable in other sectors of non-commutative field theories. A good example is provided by non-commutative solitons, classical solutions of the equations of motion which exist only because the non-commutativity scale prevents them from collapsing [17]. It has been shown recently that these solitons can travel at velocities larger than the speed of light [18, 19].

There is another facet to the issue of superluminosity. $\mathcal{N} = 4$ non-commutative Yang-Mills is the effective field theory on a D3-brane in presence of a large-NS B background field [9]. In this framework, and as explained in [20], one must distinguish between the open string $G_{\mu\nu}$ and closed string $g_{\mu\nu}$ metrics. In the Seiberg-Witten limit, which can be defined as $\alpha' B \to \infty$ for fixed g_{ij} with i, j spatial,

$$G_{ij} \sim (\alpha' B)^2 g_{ij}\,.$$

Then $g_{ij}/G_{ij} \to 0$ and the speed of light in the open string metric effectively goes to zero. Conversely, for fixed G_{ij} and $g_{ij}/G_{ij} \to 0$, the speed of light in the closed string metric effectively goes to infinity. In either case, events which from the open string metric point of view propagate faster than the speed of light are always in the future light-cone of the closed string metric and there is no *per se* superluminal propagation.[1] Nevertheless, for fields on the D-brane, there is no limit to the speed of light in the non-commutative plane. Such a phenomenon is not specific to the non-commutative gauge theory/string setting but has been discussed for instance in the context of asymmetrically warped Randall-Sundrum cosmological scenarios [21, 22].[2]

In the following sections we will extend our discussion to the fermion and gauge boson sectors of $\mathcal{N} = 4$ non-commutative U(1) Yang-Mills theory. Not surprisingly, we will find that interactions give rise again to group velocities larger than one. But we will also uncover something new. We will show that, above some critical temperature T_c, $\mathcal{N} = 4$ non-commutative U(1) Yang-Mills theory becomes thermodynamically unstable because of the appearance of a tachyonic collective mode.

3. Vector quasi-particles and tachyons

We want to calculate the photon polarization tensor at finite temperature and at one loop. This issue is technically involved already for ordinary gauge theories. Physically, this is because the polarization tensor at finite temperature encompasses many different phenomena, like Debye screening of static electric charges, generation of a thermal mass for the transverse modes, appearance of collective longitudinal excitations and the physics of Landau damping [14, 24]. These are soft phenomena, relevant on scales larger than the typical particle wavelength in the thermal bath. For soft external momenta, $p, \omega \ll T$, the leading contribution to $\Pi_{\mu\nu}(\omega, p)$ is due to hard internal momenta, $k \sim T$, and the calculations can be greatly simplified by expanding the integrand in powers of p/k. Keeping only the leading term is known in the literature as the hard thermal loop (HTL) approximation [15]. In the non-commutative case, the non-planar contribution to the polarization tensor involves three independent parameters: p, $\tilde{p}$ and T. Two of them, $1/\tilde{p}$ and the temperature T, act as competing cut-off scales. Thus the leading contribution to the non-planar diagrams comes from modes $k \sim \min(T, 1/\tilde{p})$. If we expand the integrand in powers of p/k (while keeping the full dependence on $\tilde{p}$), the condition for the first term to dominate the expansion is $p \ll \min(T, 1/\sqrt{\theta})$. The non-commutative version of the HTL approximation applies thus for external momenta smaller than both the temperature and the inverse Moyal cell radius. Our aim is to study effects associated with the UV/IR mixing characteristic of non-commutative theories. Since the HTL approximation takes into account the hard momenta circulating in loops, it will also encode the main contribution to UV/IR mixing. A non-trivial point is that in the HTL approximation the correction to the polarization tensor (in fact to all two-

[1] We thank D. Bak for discussions of this issue. (See also [18].)

[2] There has also been some work on a brane world scenario with non-commutative extra dimensions (but without gravity) [23]. In this setting we live on a soliton of non-commutative space, like in [17]. Space-time on the soliton is Poincaré invariant, but the bulk is non commutative and signals could in principle travel at arbitrary large velocities.

points functions) is gauge independent [15]. This is a well-known result in ordinary gauge theories at finite temperature [25] and its generalization to non-commutative theories is straightforward. We give an example of this in the appendix B. Let us remind that in ordinary gauge theories at zero temperature, the poles of propagators are gauge invariant quantities [26]. In non-commutative gauge theories, there is no general proof of this statement, but only checks at one-loop order [5, 27].[3] This is nevertheless sufficient to extract physical information from otherwise gauge dependent two-point functions. In particular, the poles of the resummed propagators (and their residues at the poles) are gauge invariant quantities and should correspond to physical excitations of the system at weak coupling.

The photon polarization tensor in the HTL approximation is given by

$$\Pi_{\mu\nu}(\omega,p) = 32\, g^2 \int \frac{d^3k}{(2\pi)^3} \left[\frac{dn(k)}{dk}\left(\frac{i\omega}{P.\hat{K}} - 1\right)\hat{K}_\mu \hat{K}_\nu + \frac{n(k)}{k}\left(\hat{K}_\mu\hat{K}_\nu + a_{\mu\nu}\right)\right] \times \\ \times \sin^2\frac{\tilde{p}.k}{2}\,, \tag{3.1}$$

where $\hat{K} = (-i, \hat{\mathbf{k}})$ and $P = (-\omega, \mathbf{p})$, $n = n_B + n_F$ and $a_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Note that the analytic continuation of the discrete Matsubara frequencies to real continuous external frequencies $\omega \to -ip_0$ is trivial in (3.1). In the limit of soft external momenta, (3.1) can be evaluated by expanding $\sin^2\frac{\tilde{p}.k}{2}$. The resulting expressions are presented in appendix A. In this section, we concentrate on the first non-trivial contribution at small momentum, which is $\mathcal{O}(\tilde{p}^2)$.[4] We have seen in the previous section that the most interesting modifications of the spectrum are likely to occur for very low momenta. Also, from now on, we take p to lie entirely in the non-commutative plane.

In an ordinary plasma, the longitudinal component of the gauge field can propagate due to the existence of collective charge oscillations, so-called plasmons. The longitudinal components of the polarization tensor can be written as $\Pi^L_{\mu\nu} = FP^L_{\mu\nu}$ where $P^L_{\mu\nu}$ is the longitudinal projector $(P^L)^2 = P^L$ defined by

$$P^L_{\mu\nu} = -\left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right) - P^T_{\mu\nu}$$

and P^T is the transversal projector

$$P^T_{00} = P^T_{0i} = 0\,, \qquad P^T_{ij} = \delta_{ij} - \hat{p}_i\hat{p}_j\,.$$

[3]At finite temperature, in the HTL loop approximation, the result is a bit stronger, as the whole correction to the propagator is gauge independent, thus even off mass-shell.

[4]More precisely, the leading term at small momentum is $\mathcal{O}(g^2\theta^2T^4p^2)$. Corrections come in two forms. There are higher terms in the low momentum expansion, $\mathcal{O}(g^2T^2(\theta Tp)^{2k})$, which are negligible when $\theta Tp \ll 1$. There can be also higher-loop corrections to the leading term, but power counting shows that they are $\mathcal{O}(g^{2k}\theta^2T^4p^2)$, and thus small at weak coupling.

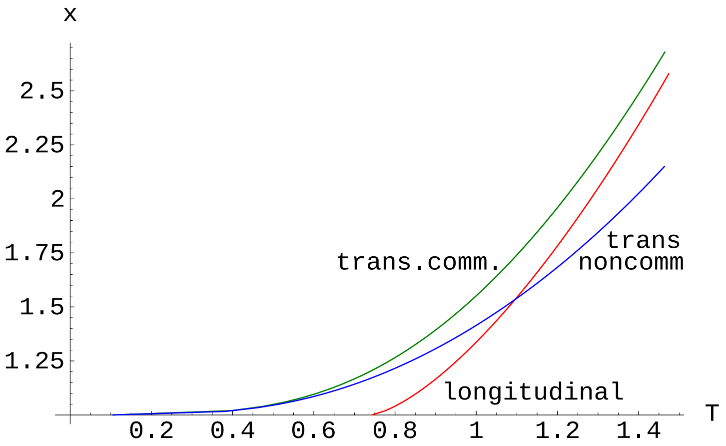

Figure 2: The figure shows the slope $x = p_0/p_{\rm nc}$ at $p_{\rm nc} = 0$ of the dispersion relation for transversal and longitudinal vector excitations as a function of T. The temperature is given in units of $1/\sqrt{g\theta}$. Longitudinal vector excitations exist only for $T \geq T_{\rm c}$.

The poles of the propagator coming from the longitudinal sector are then given by

$$P^2 - F = P^2 \left[1 + \frac{\pi^2 g^2 \theta^2 T^4}{3} \left(2 - 3x^2 + \frac{3}{2}\, x\,(x^2-1) \log \frac{x+1}{x-1} \right) \right] = 0\,, \qquad (3.2)$$

where $x = i\omega/p = p_0/p$. The solution $P^2 = 0$ is a gauge artifact due to the covariant gauge used in the calculation. The zeroes of the expression in square brackets represent the true physical excitations of the system. The function multiplying $\pi^2 g^2 \theta^2 T^4/3$ inside the square brackets is even in x, real for $x^2 > 1$ and monotonically increasing from -1 to zero for $x > 1$. Therefore there are solutions to (3.2) only provided $T \geq T_{\rm c}$, where the critical temperature is given by

$$T_{\rm c} = \left(\frac{3}{\pi^2 g^2 \theta^2} \right)^{1/4}. \qquad (3.3)$$

In a usual plasma of massless particles, the plasmon mode exists at any non-zero temperature. An unusual feature of the non-commutative plasma is that there is a critical temperature, below which the longitudinal excitations do not exist. We can shed some light on this phenomenon by thinking of the system as a gas of dipoles of size $l \sim \theta T$. On scales $p > 1/\theta T$, the system looks like a gas of charged particles. Now, ordinary plasmons exists only for soft momenta $p \lesssim gT$ [24]. When $1/\theta T < gT$ there is then a range of momenta large enough for resolving the dipoles into individual charged particles but still lower than the characteristic scale of gauge interactions, gT. In this situation, one would expect propagating longitudinal modes. On the other hand, when $1/\theta T > gT$ the system essentially behaves as a gas of neutral particles and there is no reason for the plasmons to exist. From these considerations we expect the appearance of plasmons at a temperature $T \sim 1/\sqrt{g\theta}$, which is indeed what we found from (3.2). We should emphasise that (3.2) is only valid for $p < 1/T\theta$ and the only information we can reliably derive is the group velocity at $p = 0$. In particular, for $T = T_c$, $x = 1$, and then it increases monotonically with the temperature. From this we conclude that non-commutative plasmons are massless and superluminous at the low end of the spectrum. For $T \gg T_c$, the dispersion relation of the plasmons is

$$p_0 \approx \sqrt{\frac{2}{15}\pi g\theta T^2}\, p\,. \tag{3.4}$$

At higher momenta, we expect that the plasmon spectrum behaves qualitatively as in figure 1. This is because for large momenta, $\tilde{p}\cdot k \sim p\theta T \gg 1$, the non-planar contribution to the polarization tensor is suppressed compared to the planar one. The polarization tensor reduces to

$$\Pi_{\mu\nu}(\omega, p) \approx 4g^2T^2 \int \frac{d\Omega}{4\pi}\left(\frac{i\omega}{P.\hat{K}}\hat{K}_\mu\hat{K}_\nu + \delta_{\mu 4}\delta_{\nu 4}\right).$$

For $p \ll gT$ but $p\theta T \gg 1$, the plasmon dispersion relation becomes

$$p_0^2 = \frac{3}{5}p^2 + \frac{4}{3}g^2T^2\,,$$

like in an ordinary plasma. Finally, we should notice that the Debye mass vanishes in the non-commutative plasma, $\Pi_{\mu\nu}(0,p) \to 0$ as $p \to 0$ so that a static background electric field is not screened [28].

We now turn to the dispersion relations of the transverse photons in the thermal bath. Since we take the momentum to lie entirely in the non-commutative plane, one of the transversal directions is orthogonal to it, and thus local, while the other lies on the non-commutative plane, along $\tilde{p}$. Concentrating again on the small momentum limit, the dispersion relation can be written as

$$x^2 - 1 = \frac{\pi^2 g^2\theta^2 T^4}{3}\left[\frac{c_i}{4}\left(5x^2 - 3x^4 + \frac{3}{2}x(x^2-1)^2\log\frac{x+1}{x-1}\right) - \delta_{2i}\right], \tag{3.5}$$

where $i = 1, 2$ refers to commutative and non-commutative transverse directions, respectively. The constants in (3.5) are $c_1 = 1$, $c_2 = 3$. Since non-commutativity breaks rotational invariance, the dispersion relation is different for each polarization. The r.h.s. is again an even function of x and is real for $x^2 > 1$. It is positive and monotonically decreasing for $x \in (1, \infty)$, implying that there is always a unique solution to (3.5) in that interval. For low temperatures, the dispersion relation can be approximated by

$$p_0 \approx \left(1 + \frac{\pi^2 g^2\theta^2 T^4}{12}\right) p\,, \tag{3.6}$$

for both transverse photons (it is interesting to note that (3.6) coincides with the result for the scalars modes). For high temperatures, the group velocity is different for the two polarizations,

$$p_0 \approx \sqrt{\frac{5 - c_i}{30}\pi g\theta T^2}\, p\,. \tag{3.7}$$

These modes simply correspond to the $T = 0$ transverse photon, dressed by the interactions with the thermal bath. Interestingly, they are not the only solutions of the dispersion relation (3.5). Replacing $x \to ix$ in (3.5) we get

$$x^2 + 1 = \frac{\pi^2 g^2 \theta^2 T^4}{3} \left[\frac{c_i}{4} \left(5x^2 + 3x^4 - 3x(x^2+1)^2 \arctan \frac{1}{x} \right) + \delta_{i2} \right] . \tag{3.8}$$

The first term of the r.h.s. is always negative for x real. Therefore there is no solution for the photon polarized perpendicular to the non-commutative plane. For the photon polarized along the non-commutative directions however, the term in square brackets equals one at $x = 0$ and decreases monotonically with x. This implies that for $T > T_{\rm c}$, with $T_{\rm c}$ given by (3.3), there is a purely imaginary solution to the dispersion relation (3.5). Thus we find a new transverse collective excitation with imaginary energy and, hence, tachyonic in nature. It is interesting that the critical temperature $T_{\rm c}$ above which the tachyonic solution exists equals the one for the existence of the plasmon mode. Close but strictly above $T_{\rm c}$, the tachyon dispersion relation is

$$p_0 \approx \pm i \, \frac{8}{3\pi} \, \left(\frac{T}{T_{\rm c}} - 1 \right) \, |p| \, . \tag{3.9}$$

Let us have a closer look at the polarization tensor (3.1). It can be divided in two pieces $\Pi_{\mu\nu} = \Pi^1_{\mu\nu} + \Pi^2_{\mu\nu}$. The first one is given by

$$\Pi^1_{\mu\nu} = 32\, g^2 \int \frac{d^3k}{(2\pi)^3} \frac{dn}{dk} \left(\frac{i\omega}{P.\hat{K}} \hat{K}_\mu \hat{K}_\nu + \delta_{4\mu}\delta_{4\nu} \right) \sin^2 \frac{\tilde{p}.k}{2} \, . \tag{3.10}$$

This expression is analogous to the one of an ordinary plasma, except for the $\sin \frac{\tilde{p}\cdot k}{2}$, which acts as momentum dependent form factor. This piece is responsible for the first term in the r.h.s. of (3.5). The second contribution to the polarization tensor is

$$\Pi^2_{ij} = 32\, g^2 \int \frac{d^3k}{(2\pi)^3} \left[\left(\frac{n}{k} - \frac{dn}{dk} \right) \hat{k}_i \hat{k}_j - \frac{n}{k} \delta_{ij} \right] \sin^2 \frac{\tilde{p}.k}{2} \, , \tag{3.11}$$

with $\Pi^2_{44} = \Pi^2_{4i} = 0$. This piece vanishes identically in ordinary QED or Yang-Mills theory at finite T and therefore genuinely reflects the non-commutativity of the coordinates. It can be evaluated in closed form to give

$$\Pi^2_{\mu\nu} = 4 g^2 T^2 \left(\frac{\tanh \frac{\pi \tilde{p} T}{2}}{\pi \tilde{p} T} - \frac{1}{2 \cosh^2 \frac{\pi \tilde{p} T}{2}} \right) \frac{\tilde{p}_\mu \tilde{p}_\nu}{|\tilde{p}|^2} \, . \tag{3.12}$$

Notice that $\Pi^2_{\mu\nu}$ is transverse, because $\tilde{p}_\mu p^\mu = 0$. It is directly analogous to the new contribution to the polarization tensor specific to non-commutative gauge theories and first derived in [3, 4]. This term is also at the origin of the delta term in the r.h.s. of (3.5), which is essential for the existence of the tachyon collective mode. We can not derive a closed form expression for the polarization tensor for arbitrary p. However using $\Pi^1_{ii}(p_0 = 0, p) = 0$ we observe that the dispersion relation for $p_0 = 0$ reduces to $p^2 \delta_{ij} = \Pi^2_{ij}$. This is non trivial only for the polarization along

$\tilde{p}$. For $T > T_c$ that equation has a solution at $p > 0$.[5] It is natural to assume that this additional pole belongs to the tachyonic branch. This suggests the following schematic structure of the tachyon branch: the branch starts at $p_0 = 0$ for $p = 0$; the square of the energy becomes negative as p increases until it reaches a minimum; it then increases until it reaches $p_0 = 0$ again, at a point which we will denote $\bar{p}$. For very high temperatures, we have

$$\bar{p} \approx \left(\frac{4g^2 T}{\pi\theta}\right)^{1/3} . \tag{3.13}$$

Let us investigate the behaviour of the tachyonic branch of the dispersion relation around $\bar{p}$ more closely. In the neighbourhood $p_0 = 0$ and when $p_0/p \approx 0$ we find $\Pi^1_{22} \approx i p_0 \,\mathrm{sgn}(\mathrm{Im}\, p_0) f(p, \tilde{p})$ with $f > 0$. At first order in p_0, the dispersion relation reduces to $p^2 = \Pi^2_{22} + \Pi^1_{22}$. This implies that p_0 must be imaginary and therefore

$$p^2 = \Pi^2_{22} - |p_0| f(p, \tilde{p}) . \tag{3.14}$$

Since the second term in the r.h.s. is negative there are solutions only for $p^2 < \Pi^2_{22}$. Using the arguments of footnote 5, we observe that there are no solutions to (3.14) for p bigger that $\bar{p}$! The polarization tensor has a branch cut in the complex x plane between $(-1, 1)$. This is true to all orders in the $\tilde{p}$ expansion (cf. the expressions in appendix A). Thus what happens is that the pole corresponding to the tachyonic collective mode becomes part of this branch cut for higher momenta. We have confirmed this behaviour numerically for different temperatures above the critical one using the results of appendix A (see figure 3).

In retrospect, the existence of tachyonic collective excitations could have been expected from the behaviour of non-commutative gauge theories without supersymmetry. In these cases, non-planar diagrams give rise to pole-like infrared singularities in gauge boson propagators [3, 4]. The sign of these poles depends on the relative number of fermionic and bosonic degrees of freedom in the adjoint representation. For the transverse photon polarized along $\tilde{p}$, the dispersion relation in four dimensions is

$$p_0^2 = p^2 + \frac{c}{\pi^2}\frac{g^2}{|\tilde{p}|^2} , \tag{3.15}$$

where $c = 2N_f - 2 - N_s$, with N_f and N_s, respectively the number of Majorana fermions and real scalars in the adjoint representation. Thus, if there are more bosons than fermions the infrared singularity tend to destabilize the system: $p_0^2 \to -\infty$ as $p \to 0$. Contrary if there more adjoint fermions than bosons, infrared divergences remove the lower end of the spectrum: $p_0 \to +\infty$ as $p \to 0$. In supersymmetric

[5] The slope of Π^2_{22} at the origin is T^4/T_c^4. When $T > T_c$, $p^2 - \Pi^2_{22} < 0$ at small p while it tends to $+\infty$ at high p. It is easy to show that that function has a single minimum and therefore there is a unique point $p > 0$, for $T > T_c$, such that $p^2 = \Pi^2_{22}$.

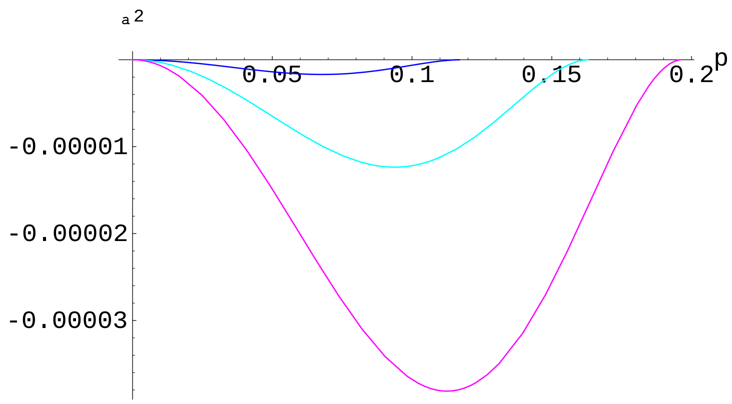

Figure 3: Dispersion relation of the tachyonic branch for different temperatures. On the vertical axes we have plotted p_0^2.

theories these infrared singularities are absent, since $c = 0$. However, because the presence of a thermal bath breaks supersymmetry, it is natural that a behaviour reminiscent of non-supersymmetric theories dominates at high temperatures. In particular as T increases, and in the static limit, we should recover the physics of a three-dimensional euclidean non-commutative gauge theory. In that limit the fermions decouple and only the zero modes of the bosonic degrees of freedom survive. Using that $\Pi^1_{ij}(p_0 = 0) = 0$, it is easy to derive the dispersion relation for the three-dimensional photon polarized along $\tilde{p}$ from our previous results. We obtain

$$p^2 - \frac{4g_3^2}{\pi}\frac{1}{|\tilde{p}|} = 0\,, \tag{3.16}$$

where the three-dimensional coupling constant is given by $g_3^2 = g^2 T$. We have checked that this coincides with the infrared leading term of the dispersion relation at one-loop for a transverse photon polarized along $\tilde{p}$ in a three-dimensional non-commutative gauge theory with seven adjoint scalars. That we only reproduce the leading term is a consequence of the HTL approximation which gives only the leading terms in a high temperature expansion. This is analogous to keeping only the UV leading terms in the Feynman integrals at zero temperature (cf. [4]). The seven adjoint scalars correspond to the zero modes of the six $\mathcal{N} = 4$ scalars plus the longitudinal mode of the gauge field. Note that at one loop the scalar modes acquire a mass of order $g_3^2 T$ in the $T \to \infty$ limit and non-zero momentum. In ordinary thermal field theories they could be integrated out to yield an effective theory consisting only of gauge bosons. In a non-commutative theory however the scalars contribute to the UV/IR mixing at the one-loop level. Thus they can not be neglected. Indeed the coefficient of (3.16) shows that the scalars do not decouple. The fermions on the other hand always decouple in the static limit, because their Matsubara mass is independent of interactions.

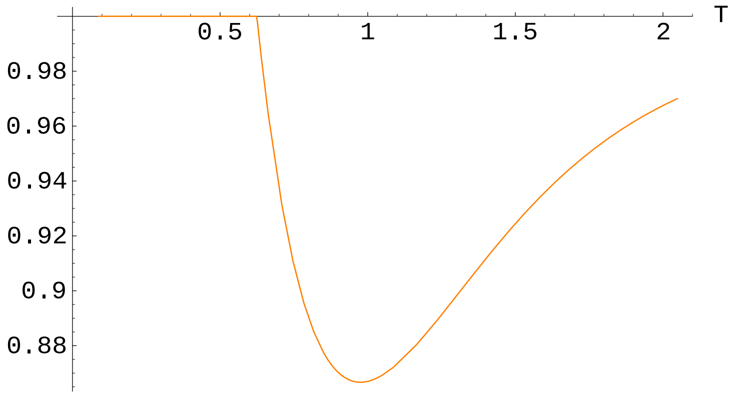

Figure 4: Sumrule for the transversal mode polarized in the non-commutative direction. The temperature is given in units of $1/\sqrt{g\theta}$. The contribution from the tachyonic pole is not included.

The presence of the tachyon mode can be alternatively detected by studying the sum rules of the theory. This will also show us that there is no other pole with $\mathrm{Im}\, p_0 \neq 0$ than the one we have found. We concentrate on the first non-trivial sum rules, which can be derived using standard techniques (see for instance [24]). For $p \ll 1/\theta T$ we have

$$\begin{aligned} \int_{-1}^{1} \frac{d\xi}{2\pi}\, \xi\, \rho_L(\xi) + \sum x_k Z_k &= \frac{2}{15}\pi^2 g^2 \theta^2 T^4\,, \\ \int_{-1}^{1} \frac{d\xi}{2\pi}\, \xi\, \rho_T^i(\xi) + \sum x_k Z_k^i &= 1\,, \end{aligned} \tag{3.17}$$

where the spectral function is $\rho_{L,T} = -2\,\mathrm{Im}(\Delta_{L,T}^{-1})$, with $\Delta_{L,T}$ denoting the dispersion relations defined in (3.2) and (3.5). The residues at the poles are given by $Z_k = (\partial\Delta_{L,T}/\partial x)^{-1}|_{x=x_k}$ and x_k denotes the position of the poles. The integrations in ξ can be thought of as integrations in p_0 by keeping p fixed. We have checked the sum rules for the longitudinal modes and the transverse modes polarized perpendicular to the non-commutative plane numerically. For the transverse modes polarized in the non-commutative plane, the sum rule ignoring the tachyonic poles is plotted in figure 4. It is saturated below the critical temperature and shows the missing contribution from the tachyonic poles above it. Taking into account the tachyonic poles also this sumrule is saturated. We leave for section 5 a discussion of the physical implications of the tachyonic excitations.

We would like to end this section by commenting on some generalizations. First let us consider the theory with U(N) gauge group. Due to the Moyal bracket interactions the U(1) component does not decouple from the SU(N) modes. The polarization tensor at one loop in the U(1) sector is now [29, 30]

$$\Pi_{\mu\nu,N} = N\,\Pi_{\mu\nu,1}\,.$$

Therefore all the results in this section apply by just substituting $g^2 \to g^2 N$. On the other hand, the polarization tensor for the SU(N) modes does not receive non-planar contributions at one loop [1, 29] and, at that order, their spectrum is that of an ordinary theory.

Another generalization is to consider theories with less supersymmetry. As long as only matter in the adjoint representation is present we expect qualitatively the same behaviour as we have found in this section. If we include also matter in the fundamental representation some differences will however appear. Matter in the fundamental representation only gives rise to planar contributions to the polarization tensor at one loop [3]. Thus, as in ordinary theories, there will be a non-vanishing Debye mass and longitudinal photon excitations for all $T > 0$. In addition, we expect the tachyonic pole in the non-commutative transverse polarization still to be present. This is because in the static limit and as $T \to \infty$ one obtains a non-commutative non-supersymmetric three-dimensional theory, with tachyonic behaviour similar to (3.16). In fact there is no new contribution from fundamental matter to the genuinely non-commutative part of the polarization tensor (3.11). Of course the overall coefficient of this term will differ depending on the content of adjoint matter, and this will change the value of the critical temperature above which the tachyon appears.

4. Fermion dispersion relations and plasmino modes

For completeness, we investigate the finite temperature self-energy of the fermions. Let us start by briefly recapitulating what happens in ordinary field theory. The most striking feature of the fermion dispersion relations at finite temperature is the appearance of a new branch of excitations. In addition to the particle-like states already present at zero temperature, there are hole-like excitations [31]. The physical picture behind this phenomenon is that the anti-fermion annihilation operator acts non-trivially on the equilibrium thermal state by creating a hole state. In the free case, the system with a hole state of momentum p would have lower energy by p than the original equilibrium thermal state and thus it is not accessible. In the interacting case however, it costs an energy which typically is $\mathcal{O}(gT)$ to create a hole and this state can have positive energy provided $p \lesssim gT$. Therefore hole states might be excited.

In the non-commutative $\mathcal{N} = 4$ gauge theory the self-energy of the fermions in the HTL approximation takes the form

$$\Sigma = \frac{8g^2}{\pi^2}\gamma_\mu \int k^3 n(k) dk \int \frac{d\Omega}{4\pi} \frac{\hat{K}^\mu}{P.\hat{K}} \sin^2 \frac{\tilde{p}.\vec{k}}{2} \,. \tag{4.1}$$

Again we concentrate on the lowest order contribution in $\tilde{p}$ from the series expansion of the sine. The inverse propagator can be written in the form

$$\gamma_\mu P^\mu - \Sigma = p\left(\Delta_+(\gamma_0 - \vec{\gamma}.\hat{p}) + \Delta_-(\gamma_0 + \vec{\gamma}.\hat{p})\right), \tag{4.2}$$

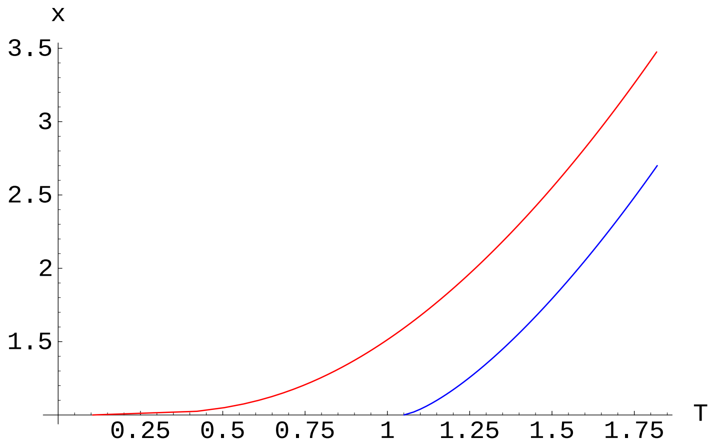

Figure 5: The slope $x = p_0/p_{\rm nc}$ at $p_{\rm nc} = 0$ for the fermionic excitations. The temperature is given in units of $1/\sqrt{g\theta}$. The plasmino mode appears only above the critical temperature $T_{\rm c}^f$.

where

$$\Delta_\pm = x \mp 1 + \frac{\pi^2 g^2\theta^2 T^4}{8}\left(x \pm \frac{1}{3} \mp (1-x^2)\left(1 - \frac{(x\mp 1)}{2}\log\frac{x+1}{x-1}\right)\right). \tag{4.3}$$

Notice that $\Delta_+(-x) = -\Delta_-(x)$. Since we are dealing with Majorana fermions, negative energy solutions can be identified with positive energy solutions. Thus we look for real positive zeroes of $\Delta_\pm$ as function of the temperature T. In this region, it turns out that $\Delta_- = 0$ has always a solution. For low temperatures, it reduces to the vacuum particle-like excitation with $p_0 = p$. On the other hand, $\Delta_+(x)$ has zeroes only for temperatures $T > T_{\rm c}^f$ with $T_{\rm c}^f = (\frac{12}{g^2\pi^2\theta^2})^{1/4}$. The situation is thus similar to the case of the plasmon mode, and the plasmino exists only at sufficiently high temperature (see figure 5).

Let us define the residues $Z_\pm^i = (\partial\Delta_\pm^{-1}/\partial x)^{-1}|_{x=x_\pm^i}$, where $x_\pm^i$ denote the zeroes of $\Delta_\pm$. We furthermore define the spectral functions $\rho_\pm = -2\,{\rm Im}(\Delta_\pm^{-1})$. One can easily prove now that the following sum rule holds

$$Z_\pm^1 + Z_\pm^2 + \int_{-1}^{1}\frac{d\xi}{2\pi}\rho_\pm(\xi) = 1\,. \tag{4.4}$$

As in the case of the vector bosons, it is implicitly understood that we keep p fixed and small and vary p_0 as we vary ξ. Let $x_\pm^1$ be the positive real zero of $\Delta_\pm$. For temperatures $T < T_{\rm c}^f$ there exists only a positive solution for Δ_-. Thus, for this low temperature region, Z_+^1 and Z_-^2 vanish in the sum rule (4.4). In the low temperature regime we find $1 > Z_-^1(Z_+^2) > 0.7263$. At very high temperatures we find $Z_\pm^1 \approx Z_\pm^2 \approx 0.5$. This is reminiscent of the behaviour of ordinary plasmino modes. Indeed the residues of particle and hole excitations in ordinary plasmas approach 0.5 at zero momentum [31].

Comparing the plasmino and plasmon critical temperatures, we observe that $T_c^f > T_c$. Since at T_c a tachyonic collective mode also appears, we expect the onset of a phase transition at that temperature. In consequence, the relevance of the plasmino mode for our system is unclear. However the qualitative properties of the fermion excitations do not depend on the presence of gauge bosons. We could have studied for example a non-commutative model with chiral superfields only and Moyal-bracket interactions. To give a specific example take a model with three chiral superfields and superpotential $W = \Phi^1 * \{\Phi^2, \Phi^3\}_*$. This corresponds to setting the $\mathcal{N} = 1$ vector field in the $\mathcal{N} = 4$ U(1) gauge theory to zero. The self-energy for the fermions is then 3/4 times the one in (4.1). The temperature where plasmino modes appear changes by a factor $(3/4)^{1/4}$.

5. Discussion

We have investigated the effects of non-commutativity on the spectrum of gauge theories. One of the most interesting phenomenon characteristic of non-commutative field theories is the mixing between UV and IR modes. In particular high momentum circulating in loops affects physics at arbitrary low energies, and generically causes the appearance of IR divergences in the perturbative expansion. We have considered a U(1) gauge theory with $\mathcal{N} = 4$ supersymmetry. In its ordinary version this theory is UV finite. Correspondingly its non-commutative deformation is free of IR divergences [8]. Finite temperature breaks supersymmetry but also provides a natural UV cut-off which allows to study the issue of UV/IR mixing in a controlled setting. Accordingly, as the temperature is increased, the effects associated with non-commutativity become more pronounced.

A first remarkable feature derived from our study of the dispersion relations is the appearance of superluminous group velocities at very soft non-commutative momenta for all degrees of freedom in the theory. This extends to fermions and vectors the results derived in [12] for scalars. Since Lorentz symmetry is broken there is no group theoretical reason for maximum velocity in the non-commutative directions. As we argued in [12], problems with causality can *a priori* be avoided in systems with space-like non-commutativity because of the existence of a preferred class of reference frames, those where time is a local coordinate (see also [18, 32]).

An interesting property of the fermion and vector dispersion relations is the appearance of collective excitations above a certain critical temperature. Hole-like excitations in the fermion sector exists for $\pi^2 g^2 \theta^2 T^4 > 12$, while collective excitations of the gauge bosons appear at a lower temperature $\pi^2 g^2 \theta^2 T^4 > 3$. These are of two types: longitudinal modes and tachyonic modes in the transverse sector polarized in the non-commutative plane. The appearance of propagating longitudinal modes can be understood by comparing with the behaviour of ordinary plasmas and picturing non-commutative particles as dipoles. In this way the very smooth UV/IR effects induced by temperature in the $\mathcal{N}=4$ theory allow to probe the spatial extent of non-commutative excitations in a neat way.

The most striking result of our work is the appearance of tachyonic collective excitations above the critical temperature $T_{\rm c}=(3/\pi^2g^2\theta^2)^{1/4}$. This implies that at $T_{\rm c}$ the system becomes thermodynamically unstable and might undergo a phase transition. To gain some understanding on the physics of this instability let us recall that non-commutative gauge theories can be alternatively described in terms of ordinary gauge fields subject to ordinary gauge transformations [20]. The Seiberg-Witten map, which relates both descriptions, is given by

$$F_{\mu\nu}=\left(\frac{1}{1-\hat{F}\theta}\hat{F}\right)_{\mu\nu}+\mathcal{O}(\partial\hat{F})\,, \tag{5.1}$$

where F and $\hat{F}$ denote respectively the ordinary and non-commutative field strengths. Using (5.1) we can define a new symplectic form, $\omega=\theta^{-1}+F$, and associate a new star product to it [33]. Thus, the Seiberg-Witten map allows to relate a configuration of the non-commutative gauge field with a modification of the Poisson structure that defines the star product. The tachyonic mode we have found appears precisely in the transverse sector polarized along a non-commutative direction. Therefore it implies a growing mode for the field strength $\hat{F}$ with indices on the non-commutative plane, i.e. $\hat{F}_{23}\neq 0$. The previous argument then suggests that the tachyonic mode corresponds to a destabilization of the initial non-commutative structure associated with $\omega=\theta^{-1}$.

A more physical picture may be obtained from string theory. The $\mathcal{N}=4$ U(1) gauge theory that we have studied can be realized as the world-volume theory on a D3-brane with an strong constant magnetic B-field or, equivalently, a D3-brane with an infinite number of delocalized D1-branes bounded to it. In particular $B=\theta^{-1}$, which implies an homogeneous distribution of D1-brane charge. The appearance of the tachyonic mode suggests that the density of D1-branes becomes inhomogeneous above $T_{\rm c}$. The new equilibrium state could correspond to a non-translational invariant distribution of D1-brane charge, analogous to the stripe phases found in [34]. Another possibility is that the phase transition at $T_{\rm c}$ is associated to the nucleation of D1-strings, dual to the evaporation of F1-strings in NCOS [35]. Although attractive, the latter possibility seems however unlikely as quite general thermodynamic arguments suggests that this may only happen at very large couplings [18, 36].

Either way, we might expect something non trivial to happen above $T_{\rm c}\sim 1/\sqrt{\theta g}$, possibly for *any* coupling. Consider the supergravity dual description of $\mathcal{N}=4$ non-commutative U(N) theories [37, 38]. The supergravity background has an interior region that tends to $AdS_5\times S^5$, while the region close to the boundary reproduces the supergravity metric associated to an infinite set of parallel D1-branes completely delocalized in two spatial directions. The value of the radial coordinate u at which the crossover between the two regimes takes place is $u_c\approx(\lambda/\theta^2)^{1/4}$, with 't Hooft coupling $\lambda = 2g^2_{\rm YM}N$ and boundary corresponding to $u \to \infty$. The field theory at finite temperature corresponds to a black hole background in the supergravity dual. It is interesting that the critical temperature $T_{\rm c} \approx 1/(g^2_{\rm YM}N\theta^2)^{1/4}$ corresponds to a black hole whose horizon is at $u \approx u_c$. For $T > T_{\rm c}$ the supergravity metric does not reduce in any region to $AdS_5 \times S^5$ and instead looks like that of smeared D1-branes along all its radial extent. The temperature $T_{\rm c} \approx 1/(g^2_{\rm YM}N\theta^2)^{1/4}$ is the critical temperature for the appearance of tachyon modes at weak coupling in $\mathcal{N} = 4$ U(N) non-commutative theory.[6] It is thus plausible that in the $T - \lambda$ plane of the phase diagram of NCSYM there is a new line of phase transitions, starting at weak coupling and possibly extending in the supergravity region. The order and nature of the phase transitions can not be established within our framework and we leave these questions for future studies.

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A. Polarization tensor and self-energy

Here we collect the results from integrating the series expansion in $\tilde{p}$ of the polarization tensor and the fermion self-energy. We assume that the momentum lies entirely in the non-commutative directions $\vec{p} = \vec{p}_{\rm nc}$. The polarization tensor can then be decomposed

$$\Pi_{\mu\nu} = F P^L_{\mu\nu} + G^c P^{T,c}_{\mu\nu} + G^{\rm nc} P^{T,{\rm nc}}_{\mu\nu} \,. \tag{A.1}$$

Here $P^L_{\mu\nu}$ is the usual longitudinal projector, $P^{T,c}_{\mu\nu}$ and $P^{T,{\rm nc}}_{\mu\nu}$ are the transverse projectors in the commutative and non-commutative directions, respectively. Terms containing $\frac{1}{P.\hat{K}} \sin^2 \frac{\tilde{p}k}{2}$ are evaluated by expanding in $|\tilde{p}|$. Using the integral representation for the hypergeometric function

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} = F(a,b,c;z) \,, \tag{A.2}$$

we find for the components of the polarization tensor of the gluons

$$F = \frac{8g^2T^2}{\pi^2} \frac{P^2|}{p_0^2} \sum_{n=1}^{\infty} A_n F\left(1, \frac{3}{2}, n + \frac{5}{2}; \frac{\vec{p}^2}{p_0^2}\right),$$

[6] Although we do not expect the supergravity dual to capture all aspects of a U(1) non-commutative gauge theory, it is interesting to note that at large temperatures, $T > T_{\rm c}$, the supergravity approximation is valid even for small N provided $g_{\rm YM} > 1$ [18].

$$G^{T,c} = \frac{8g^2T^2}{\pi^2} \sum_{n=1}^{\infty} A_n F\left(\frac{1}{2}, 1, n+\frac{5}{2}; \frac{\tilde{p}^2}{p_0^2}\right),$$

$$G^{T,\mathrm{nc}} = 2g^2T^2 \left[\frac{2\tanh\frac{|\tilde{p}|\pi T}{2}}{|\tilde{p}|\pi T} - \frac{1}{\cosh^2\frac{|\tilde{p}|\pi T}{2}}\right] + $$

$$+ \frac{8g^2T^2}{\pi^2} \sum_{n=1}^{\infty} A_n (2n+1) F\left(\frac{1}{2}, 1, n+\frac{5}{2}; \frac{\tilde{p}^2}{p_0^2}\right), \tag{A.3}$$

where we defined

$$A_n = (-)^n \frac{2n+2}{2n+3} \frac{2^{2n+2}-1}{2^{2n+1}} \zeta(2n+2) \left(|\tilde{p}|T\right)^{2n}. \tag{A.4}$$

The self-energy of the fermions can be obtained in an analogous manner.

$$\Sigma = \frac{4T^2g^2}{\pi^2} \sum_{n=1}^{\infty} B_n \left[\gamma_0 \frac{F\left(\frac{1}{2}, 1, n+\frac{3}{2}; \frac{\tilde{p}^2}{p_0^2}\right)}{p_0} - (\vec{\gamma}.\hat{p}) \frac{|\vec{p}| F\left(1, \frac{3}{2}, n+\frac{5}{2}; \frac{\tilde{p}^2}{p_0^2}\right)}{p_0^2(2n+3)}\right], \tag{A.5}$$

and

$$B_n = (-)^n \frac{2^{2n+2}-1}{2^{2n+1}} \zeta(2n+2) \left(|\tilde{p}|T\right)^{2n}. \tag{A.6}$$

B. Gauge independence

We want to show that the two-point functions of non-commutative gauge theories are gauge invariant in the HTL approximation. The argument is actually a straightforward generalization of the results of [15]. For convenience we repeated here for the simplest case of the fermion self-energy. We refer to [15] or [25] for a discussion of the gauge independence of the polarization tensor.

In a covariant gauge, the gauge-dependent contribution to the self-energy is of the form

$$\delta_\xi \Sigma(P) \sim \int_K \gamma_\mu \frac{1}{\not{P} - \not{K}} \gamma_\nu \mathcal{D}^\xi_{\mu\nu}(K) \sin^2 \frac{\tilde{p}\cdot k}{2}.$$

With $\mathcal{D}^\xi_{\mu\nu} = K_\mu K_\nu / K^4$,

$$\delta_\xi \Sigma(P) \sim \int_K \not{K}(\not{P} - \not{K})\not{K} \frac{1}{(P-K)^2} \frac{1}{K^4} \sin^2 \frac{\tilde{p}\cdot k}{2}.$$

With $\not{K}(\not{P} - \not{K})\not{K} = -(P-K)^2\not{K} + P^2\not{K} - K^2\not{P} \approx -(P-K)^2\not{K} - \not{P}K^2$, using $P \ll K$, the gauge-dependent part reduces to

$$\delta_\xi \Sigma(P) \sim -\int_K \not{P} \frac{1}{(P-K)^2} \frac{1}{K^2} \sin^2 \frac{\tilde{p}\cdot k}{2}.$$

For soft ($P \ll T$) and small ($p \ll 1/\theta T$) external momentum, this contribution is manifestly subdominant compared to the HTL contribution to the self-energy,

$$\Sigma_{\mathrm{HTL}}(P) \sim \int_K \not{K} \frac{1}{(P-K)^2} \frac{1}{K^2} \sin^2 \frac{\tilde{p} \cdot k}{2} \,.$$

For soft ($P \ll \min(T, 1/\sqrt{\theta})$) but otherwise generic $\tilde{p} \cdot k$, because $\sin^2 x \leq 1$ the (absolute value of the) integrands in both expression are bounded above by the corresponding expressions in the commutative version of the theory. Furthermore, these expressions are "in phase" for the same p and consequently the gauge dependent piece is always subleading compared to the HTL contribution.

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